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# Triality and Quantization of Singularities in Massive Fermion

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**Abstract** *It is proved that fermions can acquire the mass through the additional non-integrable exponential factor. For this propose the special vector potential associated with the spinor field was introduced. Such a vector potential has close relation with the triality property in Dirac spinors and plays crucial role in the construction of massive term. It is shown that the change in phase of a wavefunction round any closed curve with the possibility of there being singularities in our vector potential will lead to the law of quantization of physical constants including the mass. The triality properties of Dirac's spinors are studied and it leads to a double covering vector representation of Dirac spinor field. It is proved that massive Dirac equation in the bosonic representation is self-dual.*

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**Key words:** triality, bosonisation, mass quantization

## I. Introduction

The concept of phase is of great practical importance in contemporary physics. For example, the theories of superconductivity and superfluidity, the Josephson effect, holography, masers and lasers are all fundamentally based on various aspects of the concept of phase. In this paper we introduce a very special non-integrable phase in massive fermion, which is determined by positive time-like vector potential $K_\mu$. This vector potential is completely determined by spinor field and plays crucial role in the construction of massive term. Such the vector potential first appeared in the redefined wavefunction of fermion in our previous work,[1] where the conformal invariance of the Dirac equation was studied. From mathematical point of view, the existence of vector $K^\mu$ is due to the existence of triality property in Dirac spinors.

It is proved that using vector $K^\mu$, we can generate the desired massive Dirac equation from the massless Dirac equation by substituting the massive spinor $\Psi$ by massless spinor $\Psi_0$,

$$\Psi \stackrel{\mathrm{d}}{=} \Psi_0\,\mathrm{e}^{\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu} = R_0\,\mathrm{e}^{\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu} + L_0\,\mathrm{e}^{\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}\,, \tag{1}$$

such that

$$\mathrm{i}\gamma^\mu\partial_\mu\Psi_0 = 0 \tag{2}$$

identifies with Dirac equation for massive fermion $\Psi$.

In his famous monopole paper "Quantization of Singularities in Electromagnetic Field"[2] Dirac emphasized that "non-integrable phases are perfectly compatible with all the general principles of quantum mechanics and do not in any way restrict their physical interpretation". He allowed for wavefunctions with non-integrable phases and conjectured that: "*The change in phase of a wavefunction round any closed curve may be different for different wave functions by arbitrary multiplies of* $2\pi$ *and is thus not sufficiently definite to be interpreted immediately in terms of the electromagnetic field*". These correspond to single magnetic poles with their strength restricted by the relation $eq/(4\pi) = (n/2)$. For Monopole Meeting in 1981, 50 years after his first paper, Dirac has sent Abdus Salam the following message: *"I am inclined now to believe that monopoles do not exist. So many years have gone by without any encouragement from the experimental side*".

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The existence of magnetic monopole is an open question, thus in our case it means the change in $\oint mK_\mu \mathrm{d}x^\mu$ round any closed curve, with the possibility of there being singularity in $\mathrm{Re}(K_\mu)$, will lead to the law of ***quantization*** of physical constants including ***mass***.

To illustrate the geometrical nature of $K_\mu$, we will study triality property in Dirac's spinors, and it leads to the double covering vector representation of spinor field. The most interesting to physicist is that: *The massive Dirac equation in bosonic representation is self-dual.*

## II. The Behavior of Massive Term

Let $x^\mu \subset R^{(1,3)}$ be (real) coordinates of ordinary space-time, $\eta_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)$. $\Psi = [(1+\gamma_5)/2]\Psi + [(1-\gamma_5)/2]\Psi \stackrel{\mathrm{d}}{=} R + L$ is 4-component (complex) spinor, $\gamma^\mu$ are $4\times 4$ Dirac matrices and $\gamma^5 = \gamma_5 \stackrel{\mathrm{d}}{=} \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$.

***Lemma*** There exists suitable complex vector $K^\mu = K^\mu_- + K^\mu_+$ which associates with spinor $\Psi = R + L$ (and completely determined by it) such that

$$K_\mu\gamma^\mu R = K^-_\mu\gamma^\mu R = L\,, \qquad K_\mu\gamma^\mu L = K^+_\mu\gamma^\mu L = R\,, \tag{3}$$

$$\bar{R}K^*_\mu\gamma^\mu = \bar{R}K^{-*}_\mu\gamma^\mu = \bar{L}\,, \qquad \bar{L}K^*_\mu\gamma^\mu = \bar{L}K^{+*}_\mu\gamma^\mu = \bar{R}\,, \tag{4}$$

here the Dirac conjugate is defined as $\bar{R} \stackrel{\mathrm{d}}{=} (R^{*T})\gamma_0$ and $K^*_\mu$ is complex conjugation of $K_\mu$. The explicit forms of these vectors are

$$K_\mu = K^+_\mu + K^-_\mu = \frac{\bar{R}\gamma_\mu R}{2\bar{R}L} + \frac{\bar{L}\gamma_\mu L}{2\bar{L}R}\,. \tag{5}$$

They satisfy

$$K_\mu\eta^{\mu\nu}K_\nu = 1\,, \quad K^\pm_\mu\eta^{\mu\nu}K^\pm_\nu = 0\,, \quad K^\pm_\mu\eta^{\mu\nu}K^\mp_\nu = \tfrac{1}{2}\,. \tag{6}$$

***Corollary*** The massive-quadratic-two form equals the interaction-cubic-trilinear form

$$\bar{R}L = \bar{R}K_\mu\gamma^\mu R = \bar{L}K^*_\mu\gamma^\mu L\,, \qquad \bar{L}R = \bar{L}K_\mu\gamma^\mu L = \bar{R}K^*_\mu\gamma^\mu R\,. \tag{7}$$

If $\Psi = R + L$, then

$$m\bar{\Psi}\Psi = m\bar{\Psi}K_\mu\gamma^\mu\Psi \tag{8}$$

(It is like Higgs mechanism without Higgs field).

***Theorem*** Charged massive Dirac equation can be rewritten in the following "uncharged massless" forms

$$\begin{aligned} &\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)R - mL \\ &= \mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu + \mathrm{i}mK_\mu)R = [\mathrm{i}\gamma^\mu\partial_\mu R_0]\Theta^{-1} = 0\,, \end{aligned} \tag{9}$$

$$\begin{aligned} &\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)L - mR \\ &= \mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu + \mathrm{i}mK_\mu)L = [\mathrm{i}\gamma^\mu\partial_\mu L_0]\Theta^{-1} = 0\,, \end{aligned} \tag{10}$$

where $R_0 \stackrel{\mathrm{d}}{=} R\,\mathrm{e}^{-\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}$ and $\Theta = \mathrm{e}^{-\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}$. It is equivalent to

$$\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\Psi - m\Psi = [\mathrm{i}\gamma^\mu\partial_\mu\Psi_0]\Theta^{-1} = 0\,, \tag{11}$$

here $\Psi_0 \stackrel{\mathrm{d}}{=} \Psi\,\mathrm{e}^{-\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}$ satisfies the massless Dirac equation.

It is known that a charged particle must be massive. Thus in our model the mass $m$ was introduced in the same way as the charge $e$! It is mathematically beautiful and physically natural.

It is important to notice that in general $K_\mu = [\,\mathrm{Re}\,(K_\mu) + \mathrm{i}\ \mathrm{Im}\,(K_\mu)]$ is complex,

$$\begin{aligned} \mathrm{i}\ \mathrm{Im}\,(K_\mu) &= \frac{(\bar{R}\gamma_\mu R - \bar{L}\gamma_\mu L)(\bar{L}R - \bar{R}L)}{4(\bar{R}L)(\bar{L}R)} = \frac{(\bar{\Psi}\gamma_\mu\gamma^5\Psi)(\bar{\Psi}\gamma^5\Psi)}{(\bar{\Psi}\gamma_\nu\Psi)\eta^{\mu\lambda}(\bar{\Psi}\gamma_\lambda\Psi)}\,, \\ \mathrm{Re}\,(K_\mu) &= \frac{(\bar{R}\gamma_\mu R + \bar{L}\gamma_\mu L)(\bar{R}L + \bar{L}R)}{4(\bar{R}L)(\bar{L}R)} = \frac{(\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\Psi)}{(\bar{\Psi}\gamma_\nu\Psi)\eta^{\nu\lambda}(\bar{\Psi}\gamma_\lambda\Psi)}\,. \end{aligned} \tag{12}$$

Only $\mathrm{Re}\,(K_\mu)$ is associated with the phase angle, and imaginary part $\mathrm{Im}\,(K_\mu)$ is associated with the "scale factor" $\sigma = \mathrm{e}^{-m\int[\mathrm{Im}\,(K_\mu)]\,\mathrm{d}x^\mu}$ of a spinor. (We can find the similar scale change in the Weyl's early work.[3]) Furthermore

$$[\,\mathrm{Re}\,(K_\mu)][\,\mathrm{Im}\,(K^\mu)] = 0\,, \tag{13}$$

$$\bar{\Psi}[\,\mathrm{Re}\,(K_\mu)]\gamma^\mu\Psi = \bar{\Psi}\Psi\,, \qquad \bar{\Psi}[\,\mathrm{Im}\,(K_\mu)]\gamma^\mu\Psi = 0\,. \tag{14}$$

One can realize that cubic-trilinear form is independent of $\mathrm{Im}\,(K_\mu)$. Thus, if

$$\bar{\Psi} \overset{\mathrm{d}}{=} \Psi^{*T}\gamma^0\,, \qquad \bar{\bar{\Psi}}_0 \overset{\mathrm{d}}{=} \bar{\Psi}\,\mathrm{e}^{-\mathrm{i}\int[(-e)A_\mu-(-m)K_\mu]\,\mathrm{d}x^\mu}\,, \tag{15}$$

then the Lagrangian

$$\bar{\bar{\Psi}}_0\mathrm{i}\gamma^\mu\partial_\mu\Psi_0 = \bar{\Psi}\mathrm{i}\gamma^\mu(\partial_\mu\Psi - \mathrm{i}eA_\mu)\Psi - m\bar{\Psi}\Psi \tag{16}$$

is independent of the "scale factor" $\sigma = \mathrm{e}^{-m\int[\mathrm{Im}\,(K_\mu)]\,\mathrm{d}x^\mu}$, i.e., is independent of $\mathrm{Im}\,(K_\mu)$.

Sometimes it is prefer to use the Lagrangian formalism as a starting point in constructing various quantum field theories. The point of Lagrangian formalism is that it makes it easy to satisfy conformal invariance and especially to obtain Noether's conserved currents. For this purpose we introduce null-twistors which satisfy $\bar{\Phi}\Phi = 0$. In special coordinate system they can be written in the following forms

$$\Phi_+ \overset{\mathrm{d}}{=} \Big[1 - \mathrm{i}x^\mu\gamma_\mu\Big(\frac{1+\gamma_5}{2}\Big)\Big]R = \exp\Big[-\mathrm{i}x^\mu\gamma_\mu\Big(\frac{1+\gamma_5}{2}\Big)\Big]R\,, \tag{17}$$

or

$$\Phi_- \overset{\mathrm{d}}{=} \Big[1 - \mathrm{i}x^\mu\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)\Big]L = \exp\Big[-\mathrm{i}x^\mu\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)\Big]L\,. \tag{18}$$

The above additional exponential factors for $R$ and $L$ first appeared in the redefined wavefunction of electron in the work of Dirac,[4] where the wave equation in conformal space was studied. Dirac introduced physical wavefunction $\phi$, connected with conformal spinor $\psi$ by $\psi = \{1 - \mathrm{i}x^\mu\gamma_\mu[(1+\gamma_5)/2]\}\phi$. From conformally invariant wave equation for $\psi$, which involves the spin matrices, he obtained the wave equation for physical spinor $\phi$. "This equation is equivalent to the usual wave equation for electron, except for the factor $(1+\gamma_5)/2$, which introduces a degeneracy", and thus this equation was rejected by Dirac himself.

The geometry of the above null-spinors is shown clearest in terms of projective twistor space.[1,5] Considering $\Phi_+$ to be fixed and solving for real solutions $x^\mu \in M^4$ of Eq. (17) (or Eq. (18)), it turns out that a solution exists only if $\bar{\Phi}_+\Phi_+ = 0$. These solutions $x^\mu(\tau)$ (for any fixed $\Phi_+$) in real Minkowski space $M^4$ constitute a null straight line (null geodesic with parameter $\tau$), and every null straight line in Minkowski space arises in this way. So a point in Minkowski space is said to be "incident" with the null twistor. This is the so-called standard flat-space twistor correspondence.

The null-twistors are form-invariant under the following conformal transformations

$$\begin{aligned}\tilde{\Phi}_+ &= S^+\Phi_+ = \Big[1 - \mathrm{i}\tilde{x}^\mu\gamma_\mu\Big(\frac{1+\gamma_5}{2}\Big)\Big](S_0R)\,,\\ \tilde{\Phi}_- &= S^-\Phi_- = \Big[1 - \mathrm{i}\tilde{x}^\mu\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)\Big](S_0L)\,,\end{aligned} \tag{19}$$

where the operators which correspond to the above conformal transformations $S^\pm$ are

$$\begin{aligned}P_\mu^\pm &= \partial_\mu && +\gamma_\mu\Big(\frac{1\pm\gamma_5}{2}\Big),\\ M_{\mu\nu}^\pm &= (x_\nu\partial_\mu - x_\mu\partial_\nu) && -\frac{1}{4}\sigma_{\mu\nu}\,,\\ K_\mu^\pm &= 2x_\mu x^\lambda\partial_\lambda - x^2\partial_\mu && +\gamma_\mu\Big(\frac{1\mp\gamma_5}{2}\Big),\\ D^\pm &= -x^\mu\partial_\mu && \mp\frac{1}{2}\gamma_5\,,\end{aligned} \tag{20}$$

here $\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$. And the operators which correspond to the induced transformations $S_0$ are

$$\begin{aligned} P_{0\mu} &= \partial_\mu\,, \\ M_{0\mu\nu} &= (x_\nu\partial_\mu - x_\mu\partial_\nu) \quad -\tfrac{1}{4}\sigma_{\mu\nu}\,, \\ K_{0\mu} &= 2x_\mu x^\lambda\partial_\lambda - x^2\partial_\mu \quad +x^\lambda\gamma_\mu\gamma_\lambda\,, \\ D_0 &= -x^\mu\partial_\mu \quad -\tfrac{1}{2}\,. \end{aligned} \tag{21}$$

Two sets of the above operators satisfy the same commutation relations of conformal Lie algebra.

Now we are in a position to study conformal property of Dirac field.

***Theorem*** Dirac Lagrangian for massive fermion can be rewritten in the following "uncharged massless ***bosonic***" form

$$\begin{aligned} \boldsymbol{L}\sqrt{g}\,\mathrm{d}^4x &= [(\partial_\mu\bar{\bar{\Phi}}_+)g^{\mu\nu}(\partial_\nu\Phi_+) + (\partial_\mu\bar{\bar{\Phi}}_-)g^{\mu\nu}(\partial_\nu\Phi_-)]\sqrt{g}\,\mathrm{d}^4x \\ &= [\bar\Psi\mathrm{i}\gamma_\nu\eta^{\nu\mu}(\partial_\mu - \mathrm{i}eA_\mu)\Psi - 2m\bar\Psi\Psi - (\partial_\mu + \mathrm{i}eA_\mu)\bar\Psi\mathrm{i}\gamma_\nu\eta^{\nu\mu}\Psi]\,\mathrm{d}^4x\,, \end{aligned} \tag{22}$$

here $\Psi = R + L$ and

$$\Phi_+ \overset{\mathrm{d}}{=} \Big[1 - \mathrm{i}x^\mu\gamma_\mu\Big(\frac{1+\gamma_5}{2}\Big)\Big]\Big(\frac{1}{\Omega}R_0\Big)\,, \qquad R_0 \overset{\mathrm{d}}{=} R\,\mathrm{e}^{-\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}\,, \tag{23}$$

$$\Phi_- \overset{\mathrm{d}}{=} \Big[1 - \mathrm{i}x^\mu\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)\Big]\Big(\frac{1}{\Omega}L_0\Big)\,, \qquad L_0 \overset{\mathrm{d}}{=} L\,\mathrm{e}^{-\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}\,, \tag{24}$$

and

$$g^{\mu\nu} = \Omega^{-2}\eta^{\mu\nu}\,, \qquad \sqrt{g} = \Omega^4\,. \tag{25}$$

The additional factor $1/\Omega$ in $\Phi_\pm$ means that we work here with the conformal spinors of degree $-1$.[4]

It is easy to prove that the first "uncharged massless bosonic" form of the above Lagrangian implies the property of invariance under conformal transformations,[1] here the null-spinors transform in the way $\tilde\Phi_+ = S^+\Phi_+$ and $\tilde\Phi_- = S^-\Phi_-$, they are form-invariant. The representations of $S^\pm$ are linear and independent of $x^\mu$. The second "original" form of Lagrangian (22) is familiar to us and its conformal invariance is by no means apparent, but due to its equivalence to the first form, it of course must have this property. We point out that the above Lagrangian is independent of $\Omega$, this means that it is independent under conformal rescaling.

The induced conformal transformations of (physical components) $R$ and $L$ are defined by

$$\begin{aligned} \tilde\Phi_+ &= S^+\Phi_+(x,R,L,\Omega,\gamma^\mu) = \Phi_+(\tilde x,\tilde R,\tilde L,\tilde\Omega,\gamma^\mu)\,, \\ \tilde\Phi_- &= S^-\Phi_-(x,L,R,\Omega,\gamma^\mu) = \Phi_-(\tilde x,\tilde L,\tilde R,\tilde\Omega,\gamma^\mu)\,, \end{aligned} \tag{26}$$

and it is not difficult to prove that

$$\tilde\Omega = \Big|\,\frac{\partial\tilde x}{\partial x}\,\Big|^{-1/4}\Omega\,, \qquad \frac{\partial x^\nu}{\partial\tilde x^\mu}[S_0\gamma_\nu S_0^{-1}\lambda] = \gamma_\mu\,,$$

$$\tilde K_\mu = \Big|\,\frac{\partial\tilde x}{\partial x}\,\Big|^{1/4}\frac{\partial x^\nu}{\partial\tilde x^\mu}K_\nu\,, \qquad \mathrm{d}^4\tilde x = \Big|\,\frac{\partial\tilde x}{\partial x}\,\Big|\,\mathrm{d}^4x\,, \tag{27}$$

$$\tilde R = \lambda^{-1}S_0R\,\mathrm{e}^{\mathrm{i}\int(1-\lambda)[eA_\mu - mK_\mu]\,\mathrm{d}x^\mu}\,, \tag{28}$$

here $\lambda = |\partial\tilde x/\partial x|^{1/4}$. We see that the induced transformations for (physical components) $R$ and $L$ are nonlinear. $K_\mu$ is a conformal vector of degree $-1$. Recall that U(1) gauge field potential $A_\mu$ is a conformal vector of degree $-1$ too.[4]

Now there is no problem to check directly that common Lagrangian for Dirac field (i.e. the second form of Eq. (22)) is unchanged under the above induced conformal transformations (27) and (28).

It is important to notice that although we bear firmly in mind that the expressions of $R_0$ and $L_0$ include the "scale factor" $\sigma = \mathrm{e}^{-\int \mathrm{Im}\,(K_\mu)\,\mathrm{d}x^\mu}$, $\mathrm{Im}\,(K_\mu)$ has disappeared from the cubic-trilinear form (14) and from Lagrangians (16) and (22).

## III. Non-integrable Exponential Factors

Physical spinor field $\Psi$ which satisfies massive charged Dirac equation can be rewritten in the following form

$$\Psi \overset{\mathrm{d}}{=} \Psi_0\,\mathrm{e}^{\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu} = R_0\,\mathrm{e}^{\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu} + L_0\,\mathrm{e}^{\mathrm{i}\int(eA_\mu - mK_\mu)\,\mathrm{d}x^\mu}\,, \tag{29}$$

where $\Psi_0$ satisfies massless Dirac equation and

$$K_\mu = \frac{\bar{R}\gamma_\mu R}{2\bar{R}L} + \frac{\bar{L}\gamma_\mu L}{2\bar{L}R} = \frac{\bar{R}_0\gamma_\mu R_0}{2\bar{R}_0 L_0} + \frac{\bar{L}_0\gamma_\mu L_0}{2\bar{L}_0 R_0}\,. \tag{30}$$

The connection between non-integrability of phase and electromagnetic potential $A_\mu$ given here is not new, which is essentially just Weyl's principle of gauge invariance[6] in its modern form. It is also contained in the work of Ivanencko and Fock, who considered a more general kind of non-integrability based on a general theory of parallel displacement of half-vector. C.N. Yang reformulated the concept of a gauge field in an integral formalism and extended Weyl's idea to the non-Abelian general case.[7]

The non-integrable phases for the wavefunctions were also discussed by Dirac in 1931,[2] where the problem of monopole was studied. He emphasized that "non-integrable phases are perfectly compatible with all the general principles of quantum mechanics and do not in any way restrict their physical interpretation". Dirac conjectured that: "The change in phase of a wavefunction round any closed curve may be different for different wave functions by arbitrary multiplies of $2\pi$". There is the famous Dirac relation $eq/(4\pi) = (n/2)$. This means that if the quantization of electric charge (the universal unit $e$) is accepted, then the above relation is the law of ***quantization*** of the magnetic pole strength.

Notice, in our case $K^\mu = [\,\mathrm{Re}\,(K^\mu) + \mathrm{i}\ \mathrm{Im}\,(K^\mu)]$ is complex. Thus only $\mathrm{i}\int[eA_\mu - m(\,\mathrm{Re}\,(K_\mu))]\,\mathrm{d}x^\mu$ is associated with the phase-change, and imaginary part $\int \mathrm{Im}\,(K_\mu)\,\mathrm{d}x^\mu$ is associated with the scale-change.

Because of the single-valued nature of the quantum mechanical wavefunction, we naturally conjecture that:

(i) The ***phase-change*** of a wavefunction round any closed curve must be close to $2n\pi$ where $n$ is some integer, positive or negative. This integer will be a characteristic of possible singularity.

(ii) The ***scale-change*** of a wavefunction round any closed curve must be close to ***zero***. As mentioned in the previous section, the Lagrangian is independent of thus scale-change. The scale-factor can be gauged away by conformal rescaling.

This is a new (very strong) assumption, and cannot be proved, not derived. It is a conjecture of the overall consistency among all the solutions to the same equation. The existence of magnetic monopole is an open question yet, thus in our case it means that the change in $\oint mK_\mu\,\mathrm{d}x^\mu$ round any closed curve, with the possibility of there being singularity in $\mathrm{Re}\,(K_\mu)$, will lead to the law of ***quantization*** of physical constants, including ***mass***.

## IV. Illustrative Examples

**Example 1** (Plane-wave solution): The simplest solution of massless Dirac equation is that all the four components of $\Psi_0$ are constants. Thus the components of complex vector $K_\mu$ are constants too. So

$$\Psi = \Psi_0\,\mathrm{e}^{-\mathrm{i}m\int K_\mu\,\mathrm{d}x^\mu} = \Psi_0\,\mathrm{e}^{-\mathrm{i}m(K_\mu x^\mu + a)}$$

(for simplicity we take $A_\mu = 0$). The solutions of this type include plane-wave solution of free electron which is the most important solution in the quantum field theory. One realizes that

in this case, the positive time-like vector $mK_\mu$ is nothing but the ***energy-momentum*** of the massive Dirac particle. Notice, there is not singularity in $K_\mu$, thus $\oint K_\mu \mathrm{d}x^\mu = 0$.

**Example 2**: Here we use a special chiral representation of Dirac matrices. Let

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{31}$$

where $\sigma^i$ are the Pauli matrices and $\Psi^T = R_0^T + L_0^T = (R_{01}, R_{02}, L_{01}, L_{02})$. In these notations, for some physical considerations, we will study the following special solution of static massless Dirac equation,

$$\begin{aligned} R_{01} &= \frac{z}{r(x+\mathrm{i}y)} - \frac{\mathrm{i}p}{(x+\mathrm{i}y)} + q\,, \qquad & R_{02} &= \frac{1}{r}\,, \\ L_{01} &= \frac{-1}{r}\,, & L_{02} &= \frac{z}{r(x-\mathrm{i}y)} + \frac{\mathrm{i}p}{(x-\mathrm{i}y)} - q\,, \end{aligned} \tag{32}$$

here $p$ and $q$ are real constants. Thus

$$\bar{R}_0 L_0 - \bar{L}_0 R_0 = 0\,, \qquad \bar{\Psi}\Psi = \bar{\Psi}_0\Psi_0 = \bar{R}_0 L_0 + \bar{L}_0 R_0 = \frac{4q}{r}\,, \tag{33}$$

and

$$K^1 = \frac{xz}{qr(x^2+y^2)} - \frac{py}{q(x^2+y^2)}\,, \quad K^2 = \frac{yz}{qr(x^2+y^2)} + \frac{px}{q(x^2+y^2)}\,,$$
$$K^3 = \frac{2z^2 + (p^2-1)r^2}{2qr(x^2+y^2)} + \frac{q}{2}r\,. \tag{34}$$

One realizes that in this special case, $K^\mu$ are real and vary inversely proportional to the radius $r$, except for the last term in $K^3$. Moreover this vector is highly singular on the $z$-axis, and includes a “vortex part”,

$$\vec{\boldsymbol{K}} = \frac{p}{q}\left[\frac{-y}{(x^2+y^2)}, \frac{x}{(x^2+y^2)}, 0\right]. \tag{35}$$

Its form is familiar to us from the Bohm–Aharonov (solenoid) experiment.

Taking integration around closed curve $\Gamma(\tau)$ (in polar coordinates),

$$r = r_0\,, \qquad \vartheta = \vartheta_0\,, \qquad \varphi = \varphi(\tau)\,, \tag{36}$$

we have

$$m\oint_{\Gamma(\tau)} \vec{K}\cdot \mathrm{d}\vec{x} = \Big(\frac{mp}{q}\Big)2\pi = 2n\pi\,. \tag{37}$$

Because of the single-valued nature of the quantum mechanical wavefunction, $n$ must be an integer. Its numerical value is a property of singular line. The $p/q$ is determined by boundary condition. Thus we get additional quantization condition $mp/q = n$.

**Example 3**: Now let us study the following special solution of massive Dirac equation,

$$\begin{aligned} R_1 &= 0\,, & R_2 &= f\,\mathrm{e}^{\mathrm{i}(wt+vz)}\,, \\ L_1 &= \frac{-2\mathrm{i}}{m} f'\,\mathrm{e}^{\mathrm{i}(wt+vz)}\,, \qquad & L_2 &= \frac{(w-v)}{m} f\,\mathrm{e}^{\mathrm{i}(wt+vz)}\,, \end{aligned} \tag{38}$$

here $(w^2 - v^2) = m^2$ ($m$ is the mass) and $f(u)$ is the function of $u = (x + \mathrm{i}y)$, $f' = (\partial_u f)$. Thus

$$\bar{R}L - \bar{L}R = 0\,, \qquad \bar{R}L + \bar{L}R = \frac{2(v-w)}{m}\bar{f}f\,, \tag{39}$$

and

$$K^1 = \frac{-\mathrm{i}}{m}\Big(\frac{f'}{f} - \frac{\bar{f}'}{\bar{f}}\Big), \quad K^2 = \frac{1}{m}\Big(\frac{f'}{f} + \frac{\bar{f}'}{\bar{f}}\Big), \quad K^3 = \frac{2}{m(w-v)}\frac{f'}{f}\frac{\bar{f}'}{\bar{f}}\,. \tag{40}$$

The special case is $f = a(x+\mathrm{i}y)^{-\lambda}\,\mathrm{e}^{B}$, here $B(u)$ is the function of $u = (x+\mathrm{i}y)$, and $B' = \partial_\mu B$ is nonsingular. Thus

$$\bar{R}L + \bar{L}R = \frac{2(v-w)a\bar{a}}{m(\bar{u}u)^\lambda}\,\mathrm{e}^{(B+\bar{B})}\,, \tag{41}$$

$$K^1 = \frac{2\lambda y}{m(x^2+y^2)} + \frac{\mathrm{i}}{m}(\bar{B}'-B')\,, \quad K^2 = \frac{-2\lambda x}{m(x^2+y^2)} + \frac{1}{m}(\bar{B}'+B')\,,$$

$$K^3 = \frac{2}{m(w-v)}\Big(\frac{\lambda}{u}+B'\Big)\Big(\frac{\lambda}{\bar{u}}+\bar{B}'\Big). \tag{42}$$

One realizes that in this special case, $K^\mu$ are real and highly singular on the $z$-axis. They include a "vortex part", which has been discussed in the previous example. The additional quantization condition for the wavefunction of this type is

$$\lambda = n/2\,, \qquad n = 0, \pm1, \pm2, \cdots. \tag{43}$$

## V. Triality and Bosonization of Fermion

One knows that the group Spin$(2n)$ is the double covering group of the rotation group SO$(2n)$, i.e., this group has two basic half-spinor (semi-spinor) representations of degree $2^{n-1}$. A particularly interesting situation which has some relevance in physics is given when $2n = 2^{n-1}$, that is $2n = 8$. In this case Spin(8) has just three irreducible representations of degree 8 which are all real, and three representation spaces (vector space) $R$, (semi-spinor spaces) $S_+$ and $S_-$ are, remarkably, all on an equal footing. There is an extra automorphism that exchanges representations which would not be related by any symmetries for other SO$(2n)$ groups. In fact, it turns out that the extra symmetry, which is known as "triality",[8−10] relates the spinor representations of SO(8) to the vector representations. The word triality is applied to the algebraic and geometric aspects of the $\Sigma_3$ symmetry which Spin(8) has. One needs to know what three objects the symmetric group $\Sigma_3$ permutes, and the answer is that it permutes representations.

***Theorem*** (Principle of triality[8,9]) There exists an automorphism $J$ of order 3 of the vector space $A = R\times S_+\times S_-$ (dimension $= 8+16$) which has the following properties: $J$ leaves the quadratic form $\Omega$ and the cubic form $F$ invariant, and $J$ maps $R$ onto $S_+$, $S_+$ onto $S_-$, and $S_-$ onto $R$.

The law of composition in the algebra $A$ is defined in terms of the quadratic forms $\Omega$ and cubic form $F$ only. And it is clear that any automorphism of the vector space $A$ which leaves these forms invariant is an automorphism of the algebra $A$.

However physicists work in 4-dimensional Minkovski space-time $M^{1+3}$ with signature $(+,-,-,-)$. It is important to notice that the minimum dimension of gamma matrices, and thus the number of complex spinor components, depends on both dimension of space-time and ***signature*** of metric. In 4-Lorentzian dimensions, gamma matrices are (at least) $4\times4$, and thus the number of ***complex*** spinor components is four too. The quadratic-two-form in vector space is defined by means of $\eta_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)$, and the quadratic-two-form in spinor space is defined by means of Dirac conjugate spinors ($\bar\psi\psi$, here $\bar\psi = \psi^{*T}\gamma^0$). We will prove that there exists a double covering vector representation of Dirac spinors. In other words there exists an automorphism $J$ of order 3 in $A = M^{1+3}\times S_1^4\times S_2^4$ (vector space and two half-spinor spaces), which leaves quadratic form and cubic form invariant up to the sign. (In Minkowski space we are faced with spacelike and timelike vectors, thus sometimes it is convenient to use term ***pseudoscalar***, here the prefix "***pseudo***" refers to automorphism $J$.)

The situation is differ from the case of SO(8), and we propose to denominate this symmetry as a "***ding***" symmetry. (Chinese ***ding*** is an ancient vessel which has two loop handles and three legs.)

Let us first introduce trinomial unit-basis $(j^\mu, f^\alpha, \varphi^\beta)$, where the basic unit vector $j^\mu$ and

two basic unit spinors $f^\alpha$, $\varphi^\beta$ are normalized such that

$$
\begin{aligned}
&j^\mu\eta_{\mu\nu}j^\nu=-1\,, && \bar{f}f=-1\,, && \bar\varphi\varphi=1\,,\\
&j^\mu=-\bar\varphi\mathrm{i}\gamma^\mu f=\bar f\mathrm{i}\gamma^\mu\varphi\,, && f=j_\mu\mathrm{i}\gamma^\mu\varphi, && \varphi=j_\mu\mathrm{i}\gamma^\mu f\,, \qquad (44)\\
&\bar\varphi f=\bar f\varphi=0\,, && j_\mu(\bar\varphi\gamma^\mu\varphi)=j_\mu(\bar f\gamma^\mu f)=0\,, && \qquad (45)\\
&j_\mu(\bar\varphi\mathrm{i}\gamma^\mu f)=-j_\nu(\bar f\mathrm{i}\gamma^\nu\varphi)=1\,. && && \qquad (46)
\end{aligned}
$$

In addition, we can introduce another unit vector $k^\mu$ which is determined by the above trinomial unit-basis and will play very important role in our theory. That is

$$k^\mu\stackrel{\mathrm{d}}{=}\bar\varphi\gamma^\mu\varphi=\bar f\gamma^\mu f\,,\qquad k^\mu k_\mu=1\,,\qquad k^\mu j_\mu=0\,. \tag{47}$$

Furthermore we need the following algebraic properties

$$
\begin{aligned}
\tfrac12(\bar\varphi\gamma^\lambda\gamma^\nu\gamma^\mu f+\bar f\gamma^\mu\gamma^\nu\gamma^\lambda\varphi)&=\epsilon^{\mu\nu\lambda\rho}k_\rho\stackrel{\mathrm{d}}{=}\epsilon^{\mu\nu\lambda}\,,\\
\tfrac12\bar\varphi(\gamma^\mu\gamma^\nu\gamma^\lambda+\gamma^\lambda\gamma^\nu\gamma^\mu)\varphi&=t^{\mu\nu\lambda\rho}k_\rho\stackrel{\mathrm{d}}{=}t^{\mu\nu\lambda}\,,\\
\tfrac12\bar f(\gamma^\mu\gamma^\nu\gamma^\lambda+\gamma^\lambda\gamma^\nu\gamma^\mu)f&=t^{\mu\nu\lambda\rho}k_\rho\stackrel{\mathrm{d}}{=}t^{\mu\nu\lambda}\,,\\
\tfrac12\bar\varphi(\gamma^\mu\gamma^\nu\gamma^\lambda-\gamma^\lambda\gamma^\nu\gamma^\mu)\varphi&=\mathrm{i}\epsilon^{\mu\nu\lambda\rho}j_\rho\stackrel{\mathrm{d}}{=}\mathrm{i}\check\epsilon^{\mu\nu\lambda}\,,\\
\tfrac12\bar f(\gamma^\mu\gamma^\nu\gamma^\lambda-\gamma^\lambda\gamma^\nu\gamma^\mu)f&=\mathrm{i}\epsilon^{\mu\nu\lambda\rho}j_\rho\stackrel{\mathrm{d}}{=}\mathrm{i}\check\epsilon^{\mu\nu\lambda}\,,\\
\tfrac12(\bar\varphi\gamma^\lambda\gamma^\nu\gamma^\mu f-\bar f\gamma^\mu\gamma^\nu\gamma^\lambda\varphi)&=\mathrm{i}t^{\mu\nu\lambda\rho}j_\rho\stackrel{\mathrm{d}}{=}\mathrm{i}\check t^{\mu\nu\lambda}\,,
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
&\bar\varphi\gamma^\mu\gamma^\nu\varphi=-\bar f\gamma^\mu\gamma^\nu f=\eta^{\mu\nu}+\mathrm{i}\epsilon^{\mu\nu\lambda\rho}k_\lambda j_\rho\,,\\
&\bar\varphi\gamma^\mu\gamma^\nu f=\bar f\gamma^\mu\gamma^\nu\varphi=\mathrm{i}(k^\mu j^\nu-j^\mu k^\nu)\,,
\end{aligned}
\tag{49}
$$

here $\epsilon^{\mu\nu\lambda\rho}$ is the Levi–Civita symbol with the definition $\epsilon^{0123}=1$, and

$$
\begin{aligned}
\epsilon^{\mu\nu\lambda\rho}&=\tfrac{\mathrm{i}}{4}\ \mathrm{tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)\,,\\
t^{\mu\nu\lambda\rho}&\stackrel{\mathrm{d}}{=}\tfrac14\ \mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)=(\eta^{\mu\nu}\eta^{\lambda\rho}+\eta^{\mu\rho}\eta^{\nu\lambda}-\eta^{\mu\lambda}\eta^{\nu\rho})\,.
\end{aligned}
\tag{50}
$$

These objects provide the required algebraic multiplication rule on the algebra $A$, that are needed. Let

$$
\begin{aligned}
\gamma^{\mu\nu\lambda}&\stackrel{\mathrm{d}}{=}\epsilon^{\mu\nu\lambda}+\mathrm{i}t^{\mu\nu\lambda}=(-\gamma^{\lambda\nu\mu})^*\,,\\
\check\gamma^{\mu\nu\lambda}&\stackrel{\mathrm{d}}{=}\check\epsilon^{\mu\nu\lambda}+\mathrm{i}\check t^{\mu\nu\lambda}=(-\gamma^{\lambda\nu\mu})^*\,,
\end{aligned}
\tag{51}
$$

then

$$
\begin{aligned}
\gamma^{\mu\nu\sigma}\eta_{\sigma\delta}\gamma^{\lambda\rho\delta}+\gamma^{\mu\rho\sigma}\eta_{\sigma\delta}\gamma^{\lambda\nu\delta}&=\ -2\eta^{\mu\lambda}\eta^{\nu\rho}\,,\\
\check\gamma^{\mu\nu\sigma}\eta_{\sigma\delta}\check\gamma^{\lambda\rho\delta}+\check\gamma^{\mu\rho\sigma}\eta_{\sigma\delta}\check\gamma^{\lambda\nu\delta}&=\ 2\eta^{\mu\lambda}\eta^{\nu\rho}\,.
\end{aligned}
\tag{52}
$$

The most important to us is: the trinomial unit-basis $(\varphi^\beta,f^\alpha,j^\mu)$ which satisfies the above conditions exists (It can be easily verified from the special case (56)).

For better understanding geometrical and physical interpretation of complex vector $K^\mu$ introduced in the previous section, it is convenient passing from trinomial (tripartite) unit-basis $(\varphi_1^\beta,f_2^\alpha,j_3^\mu)$ to the null-basis $(r^\alpha,l^\beta,k_\pm^\mu)$, where the normalized right-handed spinor $r$ and left-handed spinor $l$ are determined in the following way

$$
\begin{aligned}
&\bar r l=\bar l r=2\,, && f=(\mathrm{i}/2)(r-l)\,, && \varphi=\tfrac12(r+l)\,,\\
&k_\pm^\mu=\tfrac12(k^\mu\pm j^\mu)\,, && k_\pm^\mu\eta_{\mu\nu}k_\pm^\nu=0\,, && k_\pm^\mu\eta_{\mu\nu}k_\mp^\nu=\tfrac12\,.
\end{aligned}
\tag{53}
$$

It is easy to prove that

$$k_\mu\gamma^\mu r=k_\mu^-\gamma^\mu r=l\,,\qquad k_\mu\gamma^\mu l=k_\mu^+\gamma^\mu l=r\,, \tag{54}$$

and it is nothing but the special case of Eq. (3).

We know that Dirac spinor includes four complex components, or equivalently $4+4=8$ independent real components. Any Dirac spinor can be decomposed in the sum of two 'half-spinors' $\Psi=\Psi_1+\Psi_2$ which can be constructed by means of $f$ and $\varphi$, i.e.,

$$\Psi_1=B^\mu\eta_{\mu\nu}\mathrm{i}\gamma^\nu f\,,\qquad \Psi_2=N^\mu\eta_{\mu\nu}\mathrm{i}\gamma^\nu\varphi\,, \tag{55}$$

here $B^\mu$ and $N^\mu$ are real vectors.

In order to easily understand our idea, it is convenient to work in the special coordinate system such that

$$[f^\alpha]^T = (0,0,\mathrm{i},0)\,, \quad [\varphi^\beta]^T = (1,0,0,0)\,, \quad j^\mu = (0,0,0,1)\,, \quad k^\mu = (1,0,0,0)\,. \tag{56}$$

(Here we use Dirac representation of gamma matrices.) In this special case

$$\Psi = \Psi_1(B) + \Psi_2(N) = \begin{pmatrix} B^3 + \mathrm{i}N^0 \\ B^1 + \mathrm{i}B^2 \\ B^0 + \mathrm{i}N^3 \\ -N^1 + \mathrm{i}N^2 \end{pmatrix}. \tag{57}$$

Let $V^\mu \in M^{1+3}$ be the vector in Minkovski space, $\Psi_1(B) \in S_1$ and $\Psi_2(N) \in S_2$ are two half-spinors referred to trinomial unit-basis defined as in the above. The quadratic two-forms and the cubic trilinear-form are well defined, i.e.,

$$V^\mu \eta_{\mu\nu} V^\nu = V_\nu V^\nu\,, \quad \bar\Psi_1 \Psi_1 = -B_\nu B^\nu\,, \quad \bar\Psi_2 \Psi_2 = N_\mu N^\mu\,, \tag{58}$$

$$V^\mu \eta_{\mu\nu} (\bar\Psi_1 \gamma^\mu \Psi_2 + \bar\Psi_2 \gamma^\mu \Psi_1) = 2(\epsilon_{\mu\nu\lambda\rho} k^\mu) V^\nu B^\lambda N^\rho\,. \tag{59}$$

Furthermore

$$B^\mu = \tfrac{1}{2}(\bar f \mathrm{i}\gamma^\mu \Psi_1 - \bar\Psi_1 \mathrm{i}\gamma^\mu f)\,, \qquad N^\mu = \tfrac{1}{2}(\bar\Psi_2 \mathrm{i}\gamma^\mu \varphi - \bar\varphi \mathrm{i}\gamma^\mu \Psi_2) \tag{60}$$

define the vector representations of half-spinors $\Psi_1$ and $\Psi_2$. One realizes that there exists a "ding" automorphism $J$ of order 3 in $M^{1+3} \times S_1 \times S_2$, which leaves quadratic two-forms and the above cubic trilinear-form invariant up to the sign. $J$ maps $M$ onto $S_1$, $S_1$ onto $S_2$, and $S_2$ onto $M$ (Or equivalently $V \to B \to N \to V$).

The most interesting for physicists is that: by passing from ordinary spinor representation to the vector representation, one can express Dirac Lagrangian in the bosonic form

$$\begin{aligned} &\tfrac{1}{2}[\bar\Psi \mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\Psi - ((\partial_\mu + \mathrm{i}eA_\mu)\bar\Psi)\mathrm{i}\gamma^\mu\Psi] - m\bar\Psi\Psi \\ &= -[(B_\nu\partial_\lambda B_\rho + N_\nu\partial_\lambda N_\rho)\check\epsilon^{\nu\lambda\rho} - (B_\nu\partial_\lambda N_\rho - N_\rho\partial_\lambda B_\nu)\check t^{\nu\lambda\rho}] \\ &\quad + m(B_\mu B^\mu - N_\nu N^\nu) + eA_\mu[(B_\nu B_\lambda + N_\nu N_\lambda)t^{\nu\mu\lambda} + 2B_\nu N_\lambda \epsilon^{\nu\mu\lambda}]\,, \end{aligned} \tag{61}$$

where $e$ and $m$ are the charge and mass of the physical particle. The Lagrangian is invariant under U(1) gauge transformation

$$\begin{aligned} &\widetilde{\Psi} = \Psi\,\mathrm{e}^{\mathrm{i}\alpha} = \Psi(\cos\alpha + \mathrm{i}\sin\alpha)\,, \qquad \widetilde{A_\mu} = A_\mu + \partial_\mu\alpha\,, \\ &\widetilde{B_\mu} = B_\mu\cos\alpha - [\check\epsilon_{\mu\nu\lambda}k^\lambda B^\nu - (k_\mu j_\nu - k_\nu j_\mu)N^\nu]\sin\alpha\,, \\ &\widetilde{N_\mu} = N_\mu\cos\alpha - [\check\epsilon_{\mu\nu\lambda}k^\lambda N^\nu + (k_\mu j_\nu - k_\nu j_\mu)B^\nu]\sin\alpha\,, \end{aligned} \tag{62}$$

where $\mathrm{e}^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\sin\alpha$.

The corresponding massive Dirac equation (which has eight independent equations) in the vector representation takes the form of

$$\begin{cases} \check\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda - \check t^{\mu\nu\lambda}\partial_\nu N_\lambda - eA_\nu(t^{\mu\nu\lambda}B_\lambda + \epsilon^{\mu\nu\lambda}N_\lambda) - mB^\mu = 0\,, \\ \check\epsilon^{\mu\nu\lambda}\partial_\nu N_\lambda + \check t^{\mu\nu\lambda}\partial_\nu B_\lambda - eA_\nu(t^{\mu\nu\lambda}N_\lambda - \epsilon^{\mu\nu\lambda}B_\lambda) + mN^\mu = 0\,, \end{cases} \tag{63}$$

or equivalently

$$\begin{cases} \partial_\mu N^\mu - [mB^\mu + eA_\nu(B_\lambda t^{\mu\nu\lambda} + N_\lambda\epsilon^{\mu\nu\lambda})]j_\mu = 0\,, \\ \partial_\mu B^\mu - [mN^\mu - eA_\nu(N_\lambda t^{\mu\nu\lambda} - B_\lambda\epsilon^{\mu\nu\lambda})]j_\mu = 0\,, \\ \nabla_\mu N_\nu - \nabla_\nu N_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}(\nabla^\lambda B^\rho - \nabla^\rho B^\lambda)\,, \end{cases} \tag{64}$$

where

$$\begin{aligned} \nabla_\mu N_\nu &\stackrel{\mathrm{d}}{=} \partial_\mu N_\nu + (m/2)\check\epsilon_{\mu\nu\rho}N^\rho + e\big(j_\mu\eta_{\nu\sigma}\epsilon^{\sigma\lambda\rho} - \tfrac{1}{2}\check\epsilon_{\mu\nu\sigma}t^{\sigma\lambda\rho}\big)A_\lambda N_\rho\,, \\ \nabla_\mu B_\nu &\stackrel{\mathrm{d}}{=} \partial_\mu B_\nu - (m/2)\check\epsilon_{\mu\nu\rho}B^\rho + e\big(j_\mu\eta_{\nu\sigma}\epsilon^{\sigma\lambda\rho} - \tfrac{1}{2}\check\epsilon_{\mu\nu\sigma}t^{\sigma\lambda\rho}\big)A_\lambda B_\rho\,. \end{aligned} \tag{65}$$

In above sense the half-spinor of the first type is dual to the half-spinor of the second type. We can define "dual" transformation $B \to N$, $N \to -B$, $m \to -m$ which leaves quadratic

form $m\bar{\Psi}\Psi = m(B_\mu B^\mu - N_\mu N^\mu)$, cubic form $A_\mu B_\nu N_\lambda \epsilon^{\mu\nu\lambda}$ and thus the above Lagrangian and Dirac equation invariant. (It looked like electro-magnetic duality which accompanied by $e \to q$ and $q \to -e$.)

If we pass from trinomial (tripartite) unit-basis $(f^\alpha, \varphi^\beta, j^\mu)$ to the null-basis $(r^\alpha, l^\beta, k^\mu_\pm)$, then the spinor can be decomposed in the sum of right-handed and left-handed spinors $\Psi = R + L$, here

$$R = \tfrac{1}{2} G^\mu \eta_{\mu\nu} \gamma^\nu l\,, \quad L = -\tfrac{1}{2} G^{*\mu} \eta_{\mu\nu} \gamma^\nu r\,, \quad G^\mu \overset{\mathrm{d}}{=} (B^\mu + \mathrm{i} N^\mu)\,. \tag{66}$$

In these notations Dirac Lagrangian takes the form of

$$L = (\partial_\mu G^*_\nu) \check{\gamma}^{\nu\mu\lambda} G_\lambda - G^*_\nu \check{\gamma}^{\nu\mu\lambda} (\partial_\mu G_\lambda) - 2\mathrm{i} e A_\mu G^*_\nu \gamma^{\nu\mu\lambda} G_\lambda + m(G^*_\nu G^{*\nu} + G_\nu G^\nu)\,. \tag{67}$$

The ordinary Dirac equation for massive fermion takes the form of

$$\check{\gamma}^{\mu\nu\lambda} \partial_\nu G_\lambda + \mathrm{i} e \gamma^{\mu\nu\lambda} A_\nu G_\lambda - m G^*_\mu = 0\,. \tag{68}$$

It can be rewritten in another self-dual form

$$\partial_\mu G^\mu - \mathrm{i} m j_\mu G^{*\mu} = 0\,, \qquad G_{\mu\nu} = (\mathrm{i}/2)\, \epsilon_{\mu\nu\lambda\rho} G^{\lambda\rho}\,, \tag{69}$$

here for simplisity we take $A_\mu = 0$ and

$$G_{\mu\nu} \overset{\mathrm{d}}{=} [(\partial_\mu G_\nu + \mathrm{i} m j_\mu G^*_\nu) - (\partial_\nu G_\mu + \mathrm{i} m j_\nu G^*_\mu)]\,. \tag{70}$$

Let us define operator $\nabla_\mu \overset{\mathrm{d}}{=} (\partial_\mu + \mathrm{i} m j_\mu \mathcal{C}^*)$, such that

$$\nabla_\mu G_{\nu\lambda} \overset{\mathrm{d}}{=} \partial_\mu G_{\nu\lambda} + \mathrm{i} m j_\mu G^*_{\nu\lambda}\,, \tag{71}$$

where $\mathcal{C}^*$ is the operator of complex conjugation: $\mathcal{C}^*\Phi = \Phi^*$. In this notation the identities

$$\nabla_\mu G_{\nu\lambda} + \nabla_\lambda G_{\mu\nu} + \nabla_\nu G_{\lambda\mu} = 0 \tag{72}$$

look like the Bianchi identities, and

$$G_{\mu\nu} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho} + G^*_{\mu\nu} \epsilon^{\mu\nu\lambda\rho} G^*_{\lambda\rho} = 2\partial_\mu [\epsilon^{\mu\nu\lambda\rho} (G_\nu G_{\lambda\rho} + G^*_\nu G^*_{\lambda\rho})] \tag{73}$$

look like the Chern–Pontryagin density and a total derivative of the Chern–Simons density. Thus the Dirac Lagrangian can be modified by the additional total derivative of the Chern–Simons density introduced in the above.

The real part $\mathrm{Re}\,(K^\mu)$ and imaginary part $\mathrm{Im}\,(K^\mu)$ which were introduced in the previous sections take the form of

$$\begin{aligned} \mathrm{Re}\,(K^\mu) &= -\pi^\mu(k) [G^{*\lambda} G^*_\lambda + G^\lambda G_\lambda] / [2(G^\nu G_\nu)(G^{*\lambda} G^*_\lambda)]\,, \\ \mathrm{i}\ \mathrm{Im}\,(K^\mu) &= -\pi^\mu_5(j) [G^{*\lambda} G^*_\lambda - G^\nu G_\nu] / [2(G^\nu G_\nu)(G^{*\lambda} G^*_\lambda)]\,, \end{aligned} \tag{74}$$

here

$$\begin{aligned} \pi^\mu(k) &= \bar{\Psi} \gamma^\mu \Psi = (t^{\nu\mu\lambda} + \mathrm{i} \epsilon^{\nu\mu\lambda}) G_\nu G^*_\lambda\,, \\ \pi^\mu_5(j) &= \bar{\Psi} \gamma^\mu \gamma^5 \Psi = (\check{t}^{\nu\mu\lambda} + \mathrm{i} \check{\epsilon}^{\nu\mu\lambda}) G_\nu G^*_\lambda\,. \end{aligned} \tag{75}$$

It means that $\mathrm{Re}\,(K^\mu)$ is associated with unit vector $k^\mu$ and $\mathrm{Im}\,(K^\mu)$ is associated with unit vector $j^\mu$. Furthermore

$$\mathrm{Re}\,(K^\mu) \eta_{\mu\nu}\ \mathrm{Im}\,(K^\nu) = k^\mu \eta_{\mu\nu} j^\nu = 0\,. \tag{76}$$

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